UK Junior Mathematical Olympiad 2010 Solutions

A1 55 $\frac{1}{1} + \frac{2}{\frac{1}{2}} + \frac{3}{\frac{1}{3}} + \frac{4}{\frac{1}{4}} + \frac{5}{\frac{1}{5}} = 1 + 2 \times 2 + 3 \times 3 + 4 \times 4 + 5 \times 5 = 1 + 4 + 9 + 16 + 25 = 55.$

A2 3 : 8 Since y is common to both ratios, we change the ratios so that $x : y = 1 : 2 = 3 : 6$ and $y : z = 3 : 4 = 6 : 8$. Then we have $x : z = 3 : 8$.

A3 14 We can note that $20^{10} = (2 \times 10)^{10} = 2^{10} \times 10^{10}$. Since $2^{10} = 1024$ has 4 digits, and multiplying by 10^{10} adds 10 zeros to the end, Tom writes down 14 digits.

A4 $\mathbf{\frac{1}{3}}$ The table shows the ways in which the monkeys (B, H and L) can select the hats. Let the hats of B, H and L be b, h and l respectively.

Monkeys			
B	H	L	
b	h	l	
b	l	h	
h	b	l	
h	l	b	*
l	h	b	
l	b	h	*

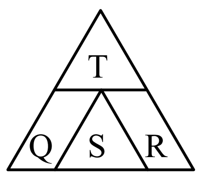
None of the monkeys have the same hat as when they arrived in only two of the six ways (*), hence the required probability is

$$\frac{2}{6} = \frac{1}{3}.$$

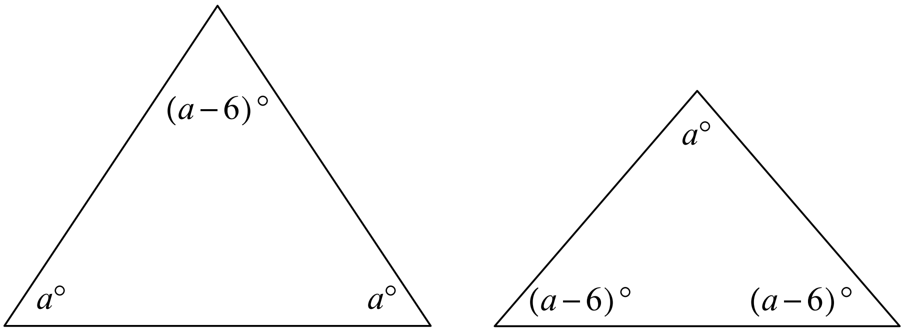
[*Alternatively*: There are $3 \times 2 \times 1 = 6$ possible ways to choose the three hats. There are two hats that B could choose. If B chose h, then L would have to choose b and H would have to choose l. If B chose l, then H would have to choose b and L would have to choose h. So once B has chosen his hat the other two are fixed. So there are just the two possible alternatives out of the six ways. So the probability is $\frac{2}{6} = \frac{1}{3}$.]

A5 2010 Let the two numbers be a and b, where $a > b$. Then we have $a + b = 97$ and $a - b = 37$. Hence $2a = 134$ and therefore $a = 67$ and $b = 30$. The product of 67 and 30 is 2010.

A6 1 : 1 Let us call the large triangle P. Since triangles T and S are congruent, they have the same height, which is half the height of P. Thus the area of each of T and S is a quarter of the area of P. Therefore parallelograms Q and R together form the other half and thus each occupies a quarter of P. So R and T are equal in area.



A7 64° Let the largest angle be $a°$, whence the smallest angle is $(a - 6)°$. There are two possibilities, shown in the diagrams below.



In the first we have $3a - 6 = 180$, so $a = 62$.
In the second we have $3a - 12 = 180$, so $a = 64$.
Thus the largest possible angle in such a triangle is 64°.

A8 13 To create a closed loop, one must use  at one end and  at the other.

Let us assume that the loop starts with  (turned this way) and ends with the , in some orientation (,  or ).
There is just 1 loop that uses only these tiles. If one tile is put between them, there are two ways in which each of the other two tiles can connect them ( or  and  or ). So there are 4 loops with three tiles.

Using all four tiles, there are two orders in which  and  can be placed, and, there are two possible orientations for each of these tiles, making $2 \times 2 \times 2 = 8$ ways in all. Hence there are $1 + 4 + 8 = 13$ possible loops altogether.

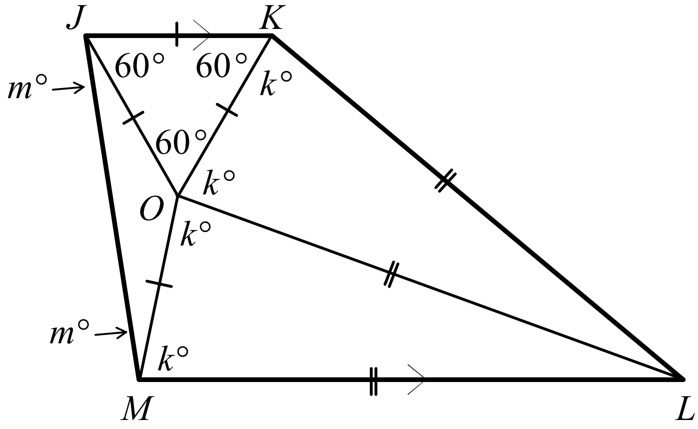
A9 2 hours Since the lowest common multiple of 6, 10 and 15 is 30, we can say that in 30 minutes Abbie writes $7 \times \frac{30}{6} = 35$ cards, Betty writes $18 \times \frac{30}{10} = 54$ cards, and Clara writes $23 \times \frac{30}{15} = 46$ cards. So together they write $35 + 54 + 46 = 135$ cards in half an hour. Thus the time taken to write 540 cards is $\frac{540}{135}$ = 4 half-hours = 2 hours.

A10 20° Since triangle JKO is equilateral, $\angle JOK = \angle KJO = \angle JKO = 60°$.

Let $\angle JMO = m°$. Then, since JMO is an isosceles triangle, $\angle MJO = m°$ and $\angle JOM = (180 - 2m)°$.

Let $\angle OKL = k°$ and so, since KLO is an isosceles triangle, $\angle LOK = k°$.

Triangles KLO and OLM are congruent (SSS), and so $\angle MOL = \angle OML = k°$.



Now taking angles at point O, we have $180 - 2m + 60 + 2k = 360$, whence $k = m + 60$.

Since JK is parallel to ML, $\angle KJM + \angle JML = 180°$ and so $(60 + m) + (m + k) = 180$.

Hence $180 = 60 + 2m + k = 60 + 2m + m + 60 = 3m + 120$, so $m = 20$, i.e. $\angle JMO = 20°$.

B1 In a sequence of six numbers, every term after the second term is the sum of the previous two terms. Also, the last term is four times the first term, and the sum of all six terms is 13.

What is the first term?

Solution

Let the first and second terms be a and b respectively. Then we derive the sequence

$$a,\ b,\ a + b,\ a + 2b,\ 2a + 3b,\ 3a + 5b.$$

We know that the last term is four times the first term, so $3a + 5b = 4a$. Therefore $a = 5b$ and so the sequence is

$$5b,\ b,\ 6b,\ 7b,\ 13b,\ 20b.$$

The sum of these is 13, so $52b = 13$, $b = \frac{13}{52} = \frac{1}{4}$ and $a = 5 \times \frac{1}{4} = \frac{5}{4}$. Thus the first term is $1\frac{1}{4}$.

B2 The eight-digit number "*ppppqqqq*", where p and q are digits, is a multiple of 45.

What are the possible values of p?

Solution

It might be argued that there is a trivial solution where $p = q = 0$. It is, however, usual to assume that numbers do not begin with zeros and so we shall proceed assuming that $p \neq 0$.

We first observe that every multiple of 45 is a multiple of both 5 and 9, and also that p and q are single-digit integers. Applying the usual rules of divisibility by 5 and 9 to the number *ppppqqqq* we deduce that $q = 0$ or $q = 5$ and that $4p + 4q$ is a multiple of 9.
In the case $q = 0$, $4p$ is a multiple of 9, hence $p = 9$.
In the case $q = 5$, $4p + 20 = 4(p + 5)$ is a multiple of 9. Therefore $p + 5$ is a multiple of 9. Hence $p = 4$.
(Thus there are two possible numbers: 99 990 000 and 44 445 555.)

B3 Jack and Jill went up a hill. They started at the same time, but Jack arrived at the top one-and-a-half hours before Jill. On the way down, Jill calculated that, if she had walked 50% faster and Jack had walked 50% slower, then they would have arrived at the top of the hill at the same time.

How long did Jill actually take to walk up to the top of the hill?

Solution

Let t be the number of hours that Jill took to the top of the hill.
So the time taken by Jack was $\left(t - 1\frac{1}{2}\right)$ hours.
If Jack had walked 50% more slowly, he would have taken twice as long, ie. $(2t - 3)$ hours.
If Jill had walked 50% faster, she would have taken $\frac{2}{3}$ of the time, ie. $\frac{2}{3}t$ hours.
So we know that $\frac{2}{3}t = 2t - 3$, whence $2t = 6t - 9$ and so $t = \frac{9}{4} = 2\frac{1}{4}$.
Hence Jill took $2\frac{1}{4}$ hours.

B4 The solution to each clue of this crossnumber is a two-digit number, not beginning with zero.

In how many different ways can the crossnumber be completed correctly?

Clues

Across
1. A triangular number
3. A triangular number

Down
1. A square number
2. A multiple of 5

1	2
3	

Solution

We start by listing the two-digit triangular numbers and two-digit square numbers:

triangular numbers:	10, 15, 21, 28, 36, 45, 55, 66, 78, 91
square numbers:	16, 25, 36, 49, 64, 81.

Since 2 Down is a multiple of 5, it ends in either 0 or 5.

Hence 3 Across ends in either 0 or 5 and there are four such triangular numbers: 10, 15, 45, and 55. In each case there is only one possible square number at 1 Down, as shown in the following figures:

8	
1	0

(a)

8	
1	5

(b)

6	
4	5

(c)

2	
5	5

(d)

Now consider 1 Across, a triangular number. In (a) and (b), there is no two-digit triangular number whose first digit is 8, and hence we can rule out cases (a) and (b).

In (c), the only triangular number whose first digit is 6 is 66. In (d), there are two triangular numbers whose first digit is 2, namely 21 and 28.

Therefore there are three different ways in which the crossnumber can be completed:

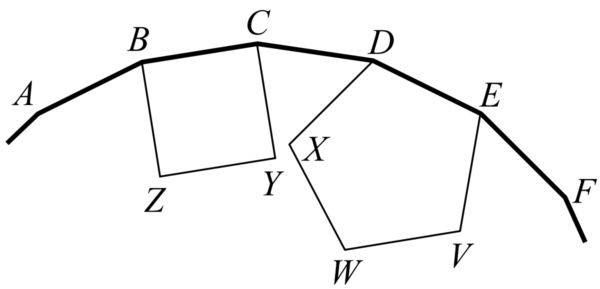
6	6
4	5

2	1
5	5

2	8
5	5

B5 The diagram shows part of a regular 20-sided polygon (an icosagon) $ABCDEF\ldots$, a square $BCYZ$ and a regular pentagon $DEVWX$.

Show that the vertex X lies on the line DY.

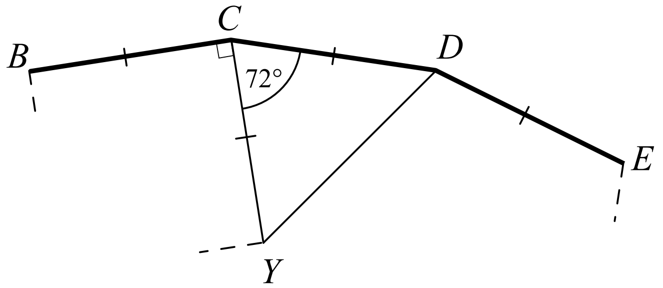


Solution

Considering the interior angles of the square, the regular pentagon, and the regular icosagon, $\angle BCY = 90°$, $\angle EDX = \left(180 - \frac{360}{5}\right)° = 108°$ and $\angle BCD = \left(180 - \frac{360}{20}\right)° = 162°$.

Now $\angle DCY = (162 - 90)° = 72°$ and also $\angle CDX = (162 - 108)° = 54°$.

Now consider triangle CDY.



Since the icosagon is regular, $BC = CD$ and, as $BCYZ$ is a square, $BC = CY$.

Therefore $CD = CY$ and CDY is an isosceles triangle.

Hence $\angle CDY = \frac{1}{2}(180 - 72)° = 54°$.

However, as observed above, $\angle CDX = 54°$ and so $\angle CDX = \angle CDY$.

Thus we can conclude that point X lies on the line DY.

B6 Sam has put sweets in five jars in such a way that no jar is empty and no two jars contain the same number of sweets. Also, any three jars contain more sweets in total than the total of the remaining two jars.

What is the smallest possible number of sweets altogether in the five jars?

Solution

Let the number of sweets in the five jars be a, b, c, d and e, where $a < b < c < d < e$. Since $d > c > b$, and b, c and d are integers, $d \geqslant b + 2$. Similarly $e \geqslant c + 2$.

Now, since any three jars contain more sweets in total than the total of the remaining two jars, in particular $a + b + c > d + e$, and so $a + b + c > b + 2 + c + 2$, hence $a > 4$.

Try $a = 5$. The smallest possible values of the other numbers are 6, 7, 8 and 9, which give a total of 35. Because 5, 6 and 7, the three smallest numbers, give a total of 18, which is over half of 35, any other selection of three of these numbers will have a total greater than that of the remaining two numbers.

Thus the smallest total is 35.